A Hybrid Finite-Element Finite-Difference Time-Domain (FE/FDTD) Technique for Solving Complex Electromagnetic Problems

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Abstract—A hybrid finite-element finite-difference time-domain (FE/FDTD) technique for solving complex electromagnetic problems is presented in this letter. The method combines the computational simplicity of the structured FDTD scheme with the versatility as well as flexibility of the finite-element method (FEM) and enables us to accurately model curved geometries and those with fine features. Numerical results that illustrate the accuracy of the method are included in the letter.

Index Terms—FDTD, FEM, hybrid techniques.

I. INTRODUCTION

IT IS WELL KNOWN that the staircasing approach employed in the conventional finite-difference time-domain (FDTD) method can introduce significant errors in the field solution when modeling curved geometries. In contrast, the finite-element method (FEM) method is well suited for handling complex geometries as it employs an unstructured mesh that conforms to these geometries. Moreover, the weak form based on the Faedo–Galerkin formulation provides a very natural way for handling the field and flux continuity conditions at the material interfaces and yields accurate results that can be further improved by using higher order interpolating functions [1]. However, unlike the FDTD, the execution of the FEM requires the solution of a matrix equation which, in turn, limits the number of unknowns because of the physical memory size of the computer. This attribute of FEM is especially exacerbating in its time-domain application, where a matrix solution is needed at each time iteration step [2], [3].

The objective of this letter is to present a hybrid scheme that combines the above two methods in a manner that retains the advantages of both [4]. However, in contrast to [4], the method presented herein does not employ an overlapping region. Consequently, it enables one to fully exploit the capabilities of modern mesh generator, and to extend the algorithm to three-dimensional cases in a convenient manner.

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II. FORMULATION

We begin by dividing $\mathcal{D}$, the computational domain, into two subdomains $\mathcal{D}_{\mathrm{FD}}$ and $\mathcal{D}_{\mathrm{FE}}$, corresponding to the FDTD and FE regions, respectively, such that $\mathcal{D} = \mathcal{D}_{\mathrm{FD}} \cup \mathcal{D}_{\mathrm{FE}}$. Next, we mesh these two regions using structured and triangular meshes, respectively, with common nodes shared at the interface but with no overlapping ($\mathcal{D}_{\mathrm{FD}} \cap \mathcal{D}_{\mathrm{FE}} = \varnothing$). The unstructured grid is confined to the vicinity of the irregular region of the domain, and this allows us to limit the number of unknowns in the FE region to a moderate size.

The FDTD solution in $\mathcal{D}_{\mathrm{FD}}$ is conventional in nature and, hence, requires no further discussion. In $\mathcal{D}_{\mathrm{FE}}$, an FE formulation with curl-conforming vector edge basis functions is employed in the time domain to discretize the second-order vector wave equation. The resulting equations are solved by using an unconditionally stable procedure that makes use of a mixed backward and central difference scheme with two stability parameters Θ_1 and Θ_2 (see [3]). It is especially crucial to employ an unconditionally stable time domain formulation for the FEM algorithm in problems with very small features that lead to large variations in the element size across the problem domain. This enables us to employ a time step which is identical to that dictated by the Courant condition in the FDTD domain, where it is typically much larger than the corresponding one in the FE domain. An additional advantage of this hybrid method is that, unlike the classic subgridding techniques in FDTD, the present approach requires no time interpolation.

Next, we turn to the time marching scheme that has been implemented in our algorithm. Let $\vec{H}_{BD}^{n-1/2}$ be the magnetic field close to the interface between FDTD and FE regions in $\mathcal{D}_{\mathrm{FD}}$, at a generic time step t^n, while $\vec{H}_{BE}^{n-1/2}$ is the adjacent magnetic field located in the $\mathcal{D}_{\mathrm{FE}}$ domain [see Fig. 1(a)]. The electric field $\vec{E}_B^n$ can be evaluated in accordance with the FDTD algorithm, provided that the location of $\vec{H}_{BE}^{n-1/2}$ coincides with that of the magnetic field in a fully structured grid. We address this problem by first identifying the FE element which contains the node point location of the desired $\vec{H}_{BE}^{n-1/2}$ and then evaluating it exactly from the knowledge of the electric fields in the element.

The time-marching scheme will now be described. We begin by updating all the electric fields $\vec{E}_{FD}^n$ in the region $\mathcal{D}_{\mathrm{FD}}$ according to the FDTD scheme, using the values of the magnetic fields $\vec{H}_{FD}^{n-1/2}$ in the region $\mathcal{D}_{\mathrm{FD}}$ together with the
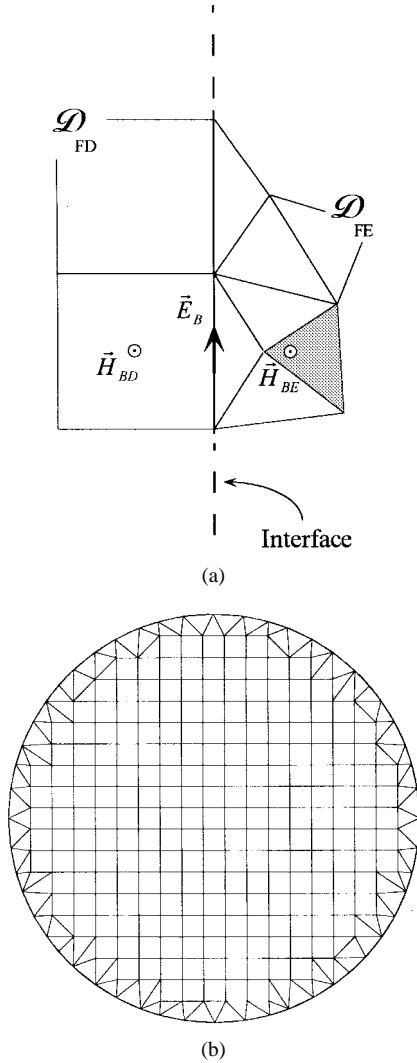
Fig. 1. (a) Treatment of the boundary conditions at the interface between the FDTD and FEM regions; (b) mesh model for the circular cavity—the triangular grid comprises 120 elements with 240 edges (68 boundary edges, 52 PEC edges) and the FDTD cell size is 1 cm.
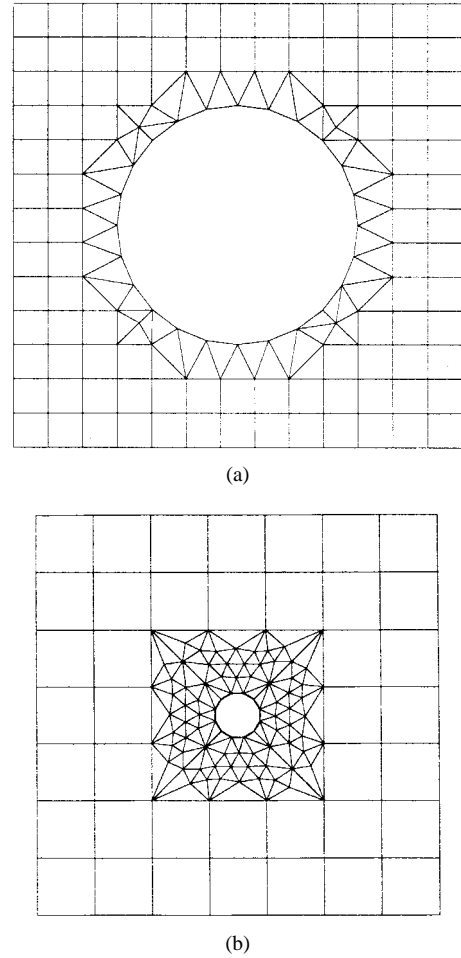

Fig. 2. (a) Mesh model for the problem of scattering from a PEC cylinder with radius $a = 3.5$ cm. The FEM mesh consists of 68 elements with 132 edges (60 external). (b) Mesh model for the wire problem; the wire has radius $a = 4$ mm. The FEM mesh now comprises 160 elements with 252 edges (24 external).

magnetic fields $\vec{H}_{BE}^{n-1/2}$ at the interface. All of the magnetic fields $\vec{H}_{FD}^{n+1/2}$ in the region $\mathcal{D}_{FD}$ can next be evaluated from $\vec{E}_{FD}^{n}$. The electric field $\vec{E}_{B}^{n}$ along the interface between $\mathcal{D}_{FD}$ and $\mathcal{D}_{FE}$ is used as a boundary condition (initial value in time, boundary value in space) for the FE procedure, so that the electric field $\vec{E}_{FE}^{n}$ at the time t^n can be evaluated throughout the region $\mathcal{D}_{FE}$. The procedure can now be repeated to continue the time iteration.

III. Numerical Results

We will now present some numerical results to validate the hybrid technique. As a first step, we have performed numerical experiments to evaluate the level of the reflections originating from the interface between the regions $\mathcal{D}_{FD}$ and $\mathcal{D}_{FE}$, and have determined that they are typically below -30 dB with respect to a fully structured FDTD scheme. However, it can rise to -15 dB at higher frequencies, especially if an electric source is placed in the immediate vicinity of the interface. It is important to note that, despite the fact that both the

FDTD and FE algorithms are fully stable, some instabilities do occur when the two methods are hybridized. In particular, spurious oscillations can appear at late times, though the occurrence of these instabilities is delayed until a later time if a smaller value is used for the time step in the FE region. The instability problems can be mitigated somewhat by using an averaging technique, although the problem cannot be fully eliminated by this procedure. However, the late time instability problem can be circumvented, and excellent numerical results can be obtained by employing the GPOF technique [5] for extrapolation to late times.

For the first example, we consider a circular cavity with a radius $r = 9.5$ cm, modeled by using the mesh shown in Fig. 1(b). The resonant frequencies of the cavity have been calculated by using both the GPOF extrapolation applied to a window of the first 200 time steps, and by employing the conventional Fourier transformation techniques. The ΔT in the FETD algorithm was chosen to equal the FDTD grid time step, i.e., ΔT was set to $2.3 \cdot 10^{-11}$ s, and the FE stability parameters were chosen to be $\Theta_1 = 0.2$ and $\Theta = 0.3$. A lower value of the time step, viz., $\Delta T = 1.2 \cdot 10^{-11}$ s, was chosen for the second experiment, 4000 time iterations were performed,
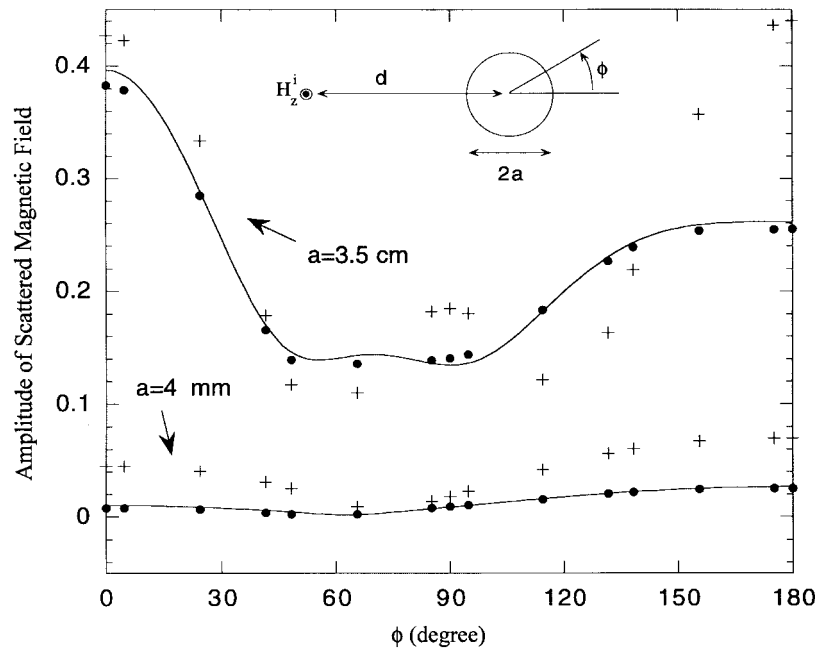
Fig. 3. Magnitude of the scattered magnetic field computed with various methods: (• • •): present hybrid technique. (+ + +): FDTD method. (———): analytically.

TABLE I
RESONANT FREQUENCIES OF THE FIRST FOUR TE MODES
OF A CIRCULAR CAVITY WITH RADIUS $r = 9.5$ cm

Analytical (MHz)	FE/FDTD (GPOF)	FE/FDTD (FFT)	FDTD
925.3	926.9	925.7	897.8
1535	1535.1	1538.6	1493
1925.8	1925.2	1920.1	1884.5
2111.5	2107.7	2108.3	2004

and 32 768 samples were Fourier transformed after zero-padding to achieve a frequency resolution of 2.54 MHz. The computed resonant frequencies of the first four TE modes are shown in Table I, together with the corresponding analytical values, and the frequencies calculated by Fourier transforming the results provided by the FDTD scheme. The improvement achieved by the hybrid method over the standalone FDTD scheme, which is susceptible to staircasing errors, is apparent from the presented results. In addition, the agreement achieved between the results obtained via the GPOF and FFT methods serves to demonstrate that the instabilities originating from the hybridization of the FE and FDTD are not significant at early times.

Next, we turn to the problem of TE scattering from perfectly conducting (PEC) circular cylinders of different radii, viz., $a = 3.5$ cm and 4 mm, the latter being well below the FDTD cell size. The discretized models of the cylinders are shown in Fig. 2. The grid size in the FDTD region is 1 cm, the total grid contains 160×160 elements, and a first-order absorbing boundary condition (ABC) is used. In both of the cases, the time step is chosen to be $\Delta T = 2.3 \cdot 10^{-11}$ s. The line source is located at a distance of 20 cm from the cylinder axis and the scattered field is observed at a distance of 12 cm. The results are shown in Fig. 3, together with the analytical values, for a

frequency of 3 GHz. Once again, the improvement achieved via the hybrid solution, relative to the FDTD scheme with staircasing, is evident from these results.

In the second case, a thin PEC wire, whose radius is well below the minimum cell size of FDTD region, was chosen to illustrate the ability of the proposed technique to handle fine features. Fig. 3 shows the amplitude of the scattered magnetic field once again, together with the analytical and FDTD results derived by modeling the wire with a single conducting cell. It is apparent that the FDTD scheme fails to yield the correct result when the radius of the wire is below the cell size, but those generated via the hybrid approach show good agreement with the analytical results. It is worthwhile mentioning that the numerical results for this problem can be significantly corrupted by the reflections from the interface, due to the extremely low values of the scattered fields. We have addressed this problem by computing the incident field from a previous run that utilizes the same mesh, but in free space, and subtracting it from the total field to derive the scattered field.

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